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REDUCTION OF THE THREE-DIMENSIONAL AXISYMMETRIC PROBLEMS OF THE THEORY OF ELASTICITY TO THE BOUNDARY VALUE PROBLEMS FOR THE ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

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Expressions for the general solution of the axisymmetric problem obtained earlier in terms of two analytic functions [1, 2] are transformed in such a manner, that only one of these functions remains under the integral sign. This also leads to the possibility of solving the axisymmetric problems by employing the methods used in the solution of the plane problem. Basically, similar transformations were employed in [3 - 5] for the particular case of a plane boundary.

A series solution for a hollow sphere with various boundary conditions at its surface is used to illustrate the method.

1. As was shown in [2], the components of the elastic displacement in an axisymmetric deformation of a solid of revolution can be written as

$$2Gw(z,r)=\frac{1}{\pi i}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)-(2z-\zeta)\varphi'(\zeta)-\psi(\zeta)]\frac{d\zeta}{g(t,\zeta)}$$

$$2Gu(z,r)=-\frac{1}{2\pi ir}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)+(2z-\zeta)\varphi'(\zeta)+\psi(\zeta)]\,g_1(t,\zeta)\,d\zeta \qquad (1.1)$$

$$g(t,\zeta)=\sqrt{(\zeta-t)(\zeta-\bar t)},\qquad g_1(t,\zeta)=(2\zeta-t-\bar t)/g(t,\zeta)$$

Here z and r are the cylindrical coordinates (z is the axis of symmetry), w, u are, respectively, the axial and radial displacements of a point, $\varkappa=3-4\nu$, ν is the Poisson ratio, G is the shear modulus, φ and ψ are analytic functions of the complex variable $\zeta=x+iy$, holomorphic in a symmetrical plane region D occupied by the meridional section of the body, x, y are rectangular coordinates lying in the plane of the above cross-section (x-axis coincides with the z-axis), and the points $t=z+ir$ and $\bar t=z-ir$ lie on this plane within D. The order of integration in (1.1) is arbitrary. The analytic functions satisfy the condition

$$\varphi(\zeta) = \overline{\varphi(\bar{\zeta})}, \qquad \psi(\zeta) = \overline{\psi(\bar{\zeta})} \tag{1.2}$$

The region D is assumed to be simply connected, and its boundary L to be a piecewise smooth symmetrical contour without cusps and composed of arcs of continuous curvature.

Assuming that the functions $\varphi(\zeta)$, $\varphi'(\zeta)$ and $\psi(\zeta)$ are continuous up to the contour L, we make t and $\bar{t}$ approach the corresponding points on the contour and perform the integration in (1.1) along the arcs of the contour contained between these two points. We write (1.1) as a single complex expression

$$2G(w - iu) = \frac{\varkappa}{\pi r}\int_{\bar{t}}^{t} \varphi(\sigma)\, U(\bar{t}, \sigma)\, d\sigma + \frac{1}{\pi r}\int_{\bar{t}}^{t} [(2z - \sigma)\varphi'(\sigma) + \psi(\sigma)]\, U(t, \sigma)\, d\sigma$$

$$U(t, \sigma) = \sqrt{(\sigma - \bar{t})/(\sigma - t)}$$

which can be transformed into

$$2G(w - iu) = S[\varkappa\overline{\varphi(\sigma)} - \bar{\sigma}\varphi'(\sigma) - \psi(\sigma)] + \int_{\bar{t}}^{t} \varphi(\sigma)\, Q(t, \sigma)\, d\sigma \tag{1.3}$$

where the points t, $\bar{t}$, σ and $\bar{\sigma}$ lie on L (here $\bar{\sigma}$ and $\bar{t}$ are regarded as functions of σ and t respectively),

$$Q(t, \sigma) = \frac{2\varkappa}{\pi|t - \bar{t}|}\left[U(\bar{t}, \sigma) - U(t, \bar{\sigma})\frac{d\bar{\sigma}}{d\sigma}\right] + \frac{2}{\pi|t - \bar{t}|}\frac{\partial}{\partial\sigma}[(\sigma + \bar{\sigma} - t - \bar{t})\, U(t, \sigma)] \tag{1.4}$$

and S is an operator on L given by

$$S(f) = -\frac{2}{\pi|t - \bar{t}|}\int_{\bar{t}}^{t} f(\sigma)\, U(t, \sigma)\, d\sigma \tag{1.5}$$

In the above transformations it was taken into account that (1.2) implies

$$\int_{\bar{t}}^{t} \overline{\varphi(\sigma)}\, U(t, \sigma)\, d\sigma = -\int_{\bar{t}}^{t} \varphi(\sigma)\, U(t, \bar{\sigma})\, d\bar{\sigma}$$

When the displacements are given on the surface of the body, Eq. (1.3) becomes an integral condition which must be satisfied by the boundary values of the analytic functions. Although the left-hand part of (1.3) is only defined for $r > 0$, we shall regard it is defined everywhere on L, assuming in accordance with (1.2) that w is even and u is odd in r.

2. Let us consider the case when the external forces are specified on the surface of the body. Altering the formulas of [2] slightly, we can express the external force intensities p_z and p_r by the boundary values of the analytic functions

$$p_z(s) = -\frac{1}{2\pi i r}\frac{d}{ds}\int_{\bar{t}}^{t} [\varphi(\sigma) - (2z - \sigma)\varphi'(\sigma) - \psi(\sigma)]\, g_1(t, \sigma)\, d\sigma$$

$$pr(s) = -\frac{1}{\pi i}\frac{d}{ds}\int_{\bar{t}}^{t} [\varphi(\sigma) + (2z - \sigma)\varphi'(\sigma) + \psi(\sigma)]\frac{d\sigma}{g(t, \sigma)} -$$

$$-\frac{1}{2\pi i r^2}\frac{dz}{ds}\int_{\bar t}^{t}[\varkappa\varphi(\sigma)+(2z-\sigma)\varphi'(\sigma)+\psi(\sigma)]\,g_1(t,\sigma)\,d\sigma \tag{2.1}$$

Here $r>0$, the points σ and t lie on the contour L, and s is the abscissa of the point t on the arc.

Let us introduce the notation

$$Z(s)=\int_0^s p_z(s')\,r'ds',\qquad R(s)=\int_0^s\left[p_r(s')+\frac{1}{r'^2}\frac{dz'}{ds'}Z(s')\right]ds' \tag{2.2}$$

where the point z_0 of intersection of L with the axis of symmetry is taken as the origin. We insert (2.1) into (2.2) and integrate with respect to s. Repeating now the transformations of Sect. 1, we arrive at

$$R+\frac{i}{r}Z=-S[\overline{\varphi(\sigma)}+\bar\sigma\varphi'(\sigma)+\psi(\sigma)]+\int_{\bar t}^{t}\varphi(\sigma)\,Q_0(t,\sigma)\,d\sigma+V(t)+C \tag{2.3}$$

Here C is a real constant, the function Q_0 is defined by (1.4) with $\varkappa=-1$, and

$$V(t)=-\frac{\varkappa+1}{\pi}\frac{t-\bar t}{|t-\bar t|}\int_{\bar t}^{t}\left[\int_{\bar t'}^{t'}\varphi(\sigma)\,g_1(t',\sigma)\,d\sigma\right]\frac{dt}{|t'-\bar t'|^2} \tag{2.4}$$

We assume R and Z to be defined everywhere on L and extend them, as even functions, to the region of negative values of r

3. We introduce the following operator on L :

$$S^{-1}(F)=\frac{1}{2}\frac{d}{d\tau}\int_{\bar\tau}^{\tau}F(t)\,U(t,\tau)\,h(t,\tau)\,dt \tag{3.1}$$

$$h(t,\tau)=\operatorname{sign}(\operatorname{Im}t\cdot\operatorname{Im}\tau)$$

The operators (1.5) and (3.1) are reciprocal to each other. Indeed, setting

$$I_1(\tau)=-\frac{1}{\pi}\int_{\bar\tau}^{\tau}\left[\int_{\bar t}^{t}f(\sigma)\,U(t,\sigma)\,d\sigma\right]U(t,\tau)\,h(t,\tau)\frac{dt}{|t'-\bar t'|^2}$$

$$I_2(t)=-\frac{1}{\pi|t-\bar t|}\int_{\bar t}^{t}\left[\int_{\bar\tau}^{\tau}F(\sigma)\,U(\sigma,\tau)\,h(\tau,\sigma)\,d\sigma\right]U(t,\tau)\,d\tau \tag{3.2}$$

and changing the order of integration, we obtain

$$I_1(\tau)=-\frac{1}{\pi}\int_{\bar\tau}^{\tau}f(\sigma)\left[\int_{l}U(t,\sigma)\,U(t,\tau)\frac{dt}{|t-\bar t|}\right]d\sigma \tag{3.3}$$

$$I_2(t)=-\frac{1}{\pi|t-\bar t|}\int_{\bar t}^{t}F(\sigma)\left[\int_{l_0}U(\sigma,\tau)\,U(t,\tau)\,d\tau\right]h(t,\sigma)\,d\sigma \tag{3.4}$$

Here $l=l(\sigma,\tau)$ denote the set of arcs $\tau\sigma_*$ and $\sigma_*\tau$, where $\sigma_*=\sigma$ if the points σ and τ lie on the same side of the axis of symmetry and $\sigma_*=\bar\sigma$ otherwise (see Fig. 1 the double line is the branch line of the radical) and $l_0=l(\sigma,t)$.

Let us find the double integral in (3.3). This is easily done by integrating along the arc

$$-\frac{1}{\pi}\int_{\sigma_*}^{\tau}U(t,\sigma)\,U(t,\tau)\frac{dt}{|t-\bar t|}-U(\bar t,\sigma)\,U(\bar t,\tau)\frac{d\bar t}{|t-\bar t|}=$$

$$= \frac{t-\bar t}{\pi\,|t-\bar t|}\left\{2\ln\left[U(\bar t,\tau)+U(\bar t,\sigma)\right]+\ln\frac{(\tau-\bar t)(\sigma-\bar t)}{t-\bar t}\right\}\Bigg|_{t=\sigma_*}^{t=\tau}=\frac{1}{2}\left[1+h(\sigma,\tau)\right]$$

When τ is replaced by ζ, the integrand function of the inner integral in (3.4) is holomorphic in the ζ-plane with a cut along l_0, and its values at the opposing edges of the cut differ only in sign. Therefore (after introducing the factor of $1/2$) any closed contour γ enclosing l_0 (Fig. 1) can be used as the contour of integration. Using the theorem of residues to perform the integration over γ we obtain

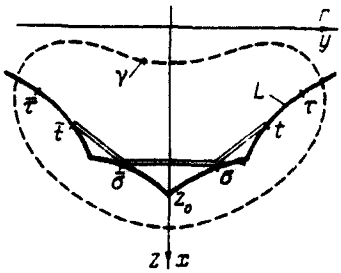

Fig. 1.

$$-\frac{1}{\pi}\int\limits_{l_0} U(\sigma,\tau)\,U(t,\tau)\,d\tau = -$$

$$\frac{i\,|t-\bar t|}{-2\pi(t-\bar t)}\int\limits_{\gamma} U(\sigma,\zeta)\,U(t,\zeta)\,d\zeta = \frac{|t-\bar t|}{2}\left(1+\frac{\sigma-\bar\sigma}{t-\bar t}\right)$$

Thus

$$I_1(\tau)=\int\limits_{z_0}^{\tau} f(\sigma)\,d\rho_\tau,$$

$$I_2(t)=\frac{1}{2}\int\limits_{\bar t}^{t} F(\sigma)\left(1+\frac{\sigma-\bar\sigma}{t-\bar t}\right)h(t,\sigma)\,d\sigma \tag{3.5}$$

Using these equations we obtain the required indentities (see also [6])

$$S^{-1}[S(f)]=\frac{d}{d\tau}I_1(\tau)=f$$

$$S[S^{-1}(F)]=\frac{d}{dt}I_2(t)+\frac{1}{2(t-\bar t)}\left(1-\frac{d\bar t}{dt}\right)[I_2(t)-I_2(t)]=F \tag{3.6}$$

The proof given here remains valid when L is a simple smooth arc without cusps, the function $f(\sigma)$ belongs to the class H^* and the function $F(t)$ is such that the operator (3.1) is an integrable function. This only requires that $F(t)$ belong to the class $H(\mu)$ when $\mu>0.5$.

4. Let us apply the operator S^{-1} to both sides of (1.3). Changing the order of intergration and taking (3.6) into account, we obtain

$$\varkappa\overline{\varphi(\tau)}-\bar\tau\varphi'(\tau)-\psi(\tau)+\frac{d}{d\tau}\int\limits_{\bar\tau}^{\tau}\varphi(\sigma)\,K(\tau,\sigma)\,d\sigma=2GS^{-1}(w-iu) \tag{4.1}$$

$$K(\tau,\sigma)=\frac{1}{2}\int\limits_{\bar t}Q(t,\sigma)\,U(t,\tau)\,dt \tag{4.2}$$

where both the kernel K and its derivative with respect to τ are continuous on any smooth part of L except at the point $\sigma=z_0$ where an integral singularity may appear. The values of $K(\tau,\tau)$ and $K(\tau,\bar\tau)$ are easily calculated, and the last term of (4.1) can then be written in the form

$$\frac{d}{d\tau}\int\limits_{\bar\tau}^{\tau}\varphi K\,d\sigma=-\frac{1}{2}\varphi(\tau)\left(1+\frac{d\bar\tau}{d\tau}\right)-\varkappa\overline{\varphi(\tau)}\left(1-i\frac{d\bar\tau}{ds}\right)+\int\limits_{\bar\tau}^{\tau}\varphi\,\frac{\partial K}{\partial\tau}\,d\sigma \tag{4.3}$$

Similarly, starting from (2.3) we obtain

$$\overline{\varphi(\tau)}+\bar\tau\varphi'(\tau)+\psi(\tau)-\frac{d}{d\tau}\int\limits_{\bar\tau}^{\tau}\varphi(\sigma)\,K_0(\tau,\sigma)\,d\sigma+\int\limits_{\bar\tau}^{\tau}\varphi(\sigma)\,K_1(\tau,\sigma)\,d\sigma=$$

$$=-S^{-1}\left(R+\frac{i}{r}Z\right)+C \tag{4.4}$$

$$K_1(\tau, \sigma) = \frac{\varkappa + 1}{2\pi} \int_l \frac{g_1(t, \tau)\, g_1(t, \sigma)}{(t - \bar{t})\,|t - \bar{t}|}\, dt \tag{4.5}$$

where K_0 is given by (4.2) in which Q is replaced by Q_0 (and $\varkappa$ by -1).

In the course of derivation of (4.4) it was assumed that $V(t)$ is a real function satisfying (1.2), and the following transformation was performed

$$S^{-1}(V) = -\frac{1}{4} \int_{\bar{\tau}}^{\tau} \frac{dV}{dt}\, g_1(t, \tau)\, h(\tau, t)\, dt \tag{4.6}$$

after which the value of V given by (2.4) was inserted and the order of integration changed.

Equations (4.1) and (4.4) are either equivalent to (1.3) and (2.3), or can be transformed into the latter by means of the operator S. They express the conditions which must be satisfied by the boundary values of the analytic functions in the case of the first and second fundamental axisymmetric problem of the theory of elasticity. Neglecting the integrals in the left-hand side gives equations expressing the boundary conditions imposable on φ and ψ in the plane case.

The function ψ does not appear under the integral sign. This opens a possibility of employing, in the course of solving the axisymmetric problem, the methods utilized in solution of the plane case (expanding into power series, conformal mapping, reduction to the Muskhelishvili-Sherman integral equations, etc.).

In the case of a plane boundary we have $K = K_0 = K_1 \equiv 0$, and the operators (1.5) and (3.1) can easily be reduced to the operators used [3 - 5] in establishing the connection between the axisymmetric and the plane state.

When the boundary is spherical, we have the following expression for the kernels

$$\frac{\partial K}{\partial \tau} = \frac{(2\varkappa + 1)\,\tau + \bar{\sigma}}{4\tau \sqrt{\sigma\tau}}, \quad \frac{\partial K_0}{\partial \tau} - K_1 = \frac{(2\varkappa + 1)\,\tau + \bar{\sigma}}{4\tau \sqrt{\sigma\tau}} - \frac{\varkappa + 1}{2\,|\tau|}\,[1 - h(\tau, \sigma)] \tag{4.7}$$

5. The above reasoning can also be applied to the solids of revolution containing internal cavities but nevertheless simply connected, although the region D itself is multiply connected. Its boundary L is composed of the outer contour L_0 and the inner contours L_k $(k = 1, 2, \ldots n)$, which will be numbered in the order in which they intersect the axis of symmetry.

The function φ (or ψ) holomorphic in D can be written in the form of a sum of the functions φ_k (or ψ_k) $(k = 0, 1, \ldots, n)$, where φ_0 and ψ_0 are holomorphic everywhere inside L_0, the functions φ_k and ψ_k $(k \geqslant 1)$ are holomorphic everywhere outside the corresponding contour L_k, and vanish at infinity. Moreover,

$$\lim_{\zeta \to \infty} \zeta\,[(\varkappa + 1)\,\varphi_k(\zeta) + \psi_k(\zeta)] = 0 \qquad (k = 1, 2, \ldots n) \tag{5.1}$$

The line of integration in (1.1) can intersect the axis of symmetry at any point within D. However the change of position of this point is accompanied by a change in the form of φ and ψ. In particular, if the line of integration intersects the axis of symmetry at a point lying between the contours L_j and L_{j+1}, then it should be assumed in all formulas that

$$\varphi(\zeta) = \varphi_0(\zeta) + \sum_{k=1}^{n} \varphi_k(\zeta)\, \mathrm{sign}\,(k - j - 0.5)$$

$$\psi(\zeta) = \psi_0(\zeta) + \sum_{k=1}^{n} \psi_k(\zeta)\,\mathrm{sign}(k - j - 0.5) \tag{5.2}$$

Under this condition, (4.1) and (4.4) hold separately for each of the contours L_k

6. Consider the axisymmetric problem for a hollow ealstic sphere bounded by the surfaces $\rho = \rho_1$ and $\rho = \rho_2$ (Fig. 2), on which either the displacements

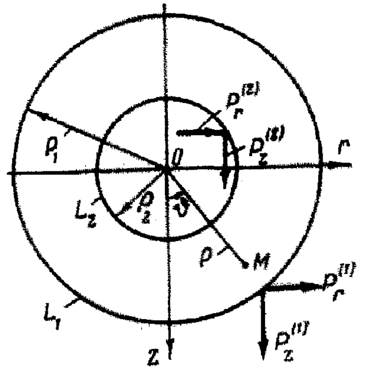

Fig. 2.

$$2Gw^{(j)} = \sum_{n=0}^{\infty} A_n^{(j)} P_n(\mu),$$

$$2Gu^{(j)} = \sqrt{1-\mu^2} \sum_{n=1}^{\infty} B_n^{(j)} P_n'(\mu) \qquad (j = 1; 2) \tag{6.1}$$

or the external forces

$$p_z^{(j)} = \frac{(-1)^{(j)}}{\rho_j} \sum_{n=0}^{\infty} C_n^{(j)} P_n(\mu),$$

$$p_r^{(j)} = (-1)^j \frac{\sqrt{1-\mu^2}}{\rho_j} \sum_{n=1}^{\infty} D_n^{(j)} P_n'(\mu) \tag{6.2}$$

$$(\mu = \cos\vartheta)$$

where $P_n(\mu)$ are the Legendre polynomials and ϑ is the polas angle, are given. In the latter case we use the formula (2.2) to obtain

$$Z^{(j)} = (-1)^{j+1} Z_0^{(j)} \frac{1-\mu}{2} + \rho_j (1 - \mu^2) \sum_{n=1}^{\infty} B_n^{(j)} P_n'(\mu)$$

$$R^{(j)} = (-1)^{j+1} \frac{1}{2\rho_j} Z_0^{(j)} \ln\frac{1+\mu}{2} + \sum_{n=0} A_n^{(j)} P_n(\mu) \tag{6.3}$$

where $2\pi Z_0^{(j)}$ is the resultant of the forces applied to the corresponding surface,

$$A_n^{(j)} = C_n^{(j)} - B_n^{(j)}, \qquad B_n^{(j)} = \frac{D_n^{(j)}}{n(n+1)} \qquad (n \geqslant 1)$$

$$A_0^{(j)} = -\sum_{n=1}^{\infty} A_n^{(j)}, \qquad Z_0^{(j)} = 2\rho_j C_0^{(j)} \qquad (j = 1, 2) \tag{6.4}$$

Inserting (6.1) and (6.3) into (4.1) - (4.4) we obtain the resulting equations in the form

$$\lambda_j \overline{\varphi(\tau_j)} \frac{\bar{\tau}_j}{\rho_j} - \bar{\tau}_j \varphi'(\tau_j) - \psi(\tau_j) - \frac{1}{2}\varphi(\tau_j)\left(1 - \frac{\bar{\tau}_j^2}{\rho_j^2}\right) + \int_{\bar{\tau}_j}^{\tau_j} \varphi K_* d\sigma_j =$$

$$= (-1)^{j+1} \frac{\varkappa - \lambda_j}{2\rho_j(\varkappa + 1)} Z_0^{(j)} \ln\frac{\tau_j}{\rho_j} + \sum_{n=-\infty}^{\infty} \frac{n+1}{2n+1} F_n^{(j)} \tau_j^n \qquad (j = 1; 2) \tag{6.5}$$

$$K_*(\tau_j \sigma_j) = \frac{(2\varkappa+1)\tau_j + \bar{\sigma}_j}{4\tau_j\sqrt{\sigma_j \tau_j}} - \frac{\varkappa - \lambda_j}{2\rho_j}[1 - h(\tau_j, \sigma_j)]$$

$$F_n^{(j)} = (A_n^{(j)} - nB_n^{(j)})\rho_j^{-n} \qquad (n \neq 0), \qquad F_0^{(j)} = A_0^{(j)} - (\varkappa - \lambda_j) C^{(j)} \tag{6.6}$$

Here $C_n^{(j)}$ is a constant. For the negative values of n we must put $A_n = A_{-n-1}$ and $B_n = B_{-n-1}$.

Equation (6.5) is valid for both, the surface with prescribed displacements ($\lambda_j = \varkappa$) and the surface with prescribed external forces ($\lambda_j = -1$).

We seek the analytic functions in the form of series

$$\varphi(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n, \qquad \psi(\zeta) = \sum_{n=-\infty}^{\infty} b_n \zeta^n \tag{6.7}$$

The coefficients a_n and b_n are real and

$$\int_{\bar{\tau}_j}^{\tau_j} \varphi K_* d\sigma_j = \left(\varkappa + \frac{1}{2}\right) \sum_{n=-\infty}^{\infty} \frac{a_n \tau_j^n}{2n+1} + \frac{1}{2}\rho_j^2 \sum_{n=-\infty}^{\infty} \frac{a_n \tau_j^{n-2}}{2n-1} - \frac{1}{2\rho_j} \sum_{n=-\infty}^{\infty} \left(\frac{2\varkappa+1}{2n+1} + \right.$$

$$\left. + \frac{1}{2n-1}\right) a_n \bar{\tau}_j^{n+1} + \frac{\varkappa - \lambda_j}{\rho_j} \left(\sum_{\substack{n=-\infty \\ (n \neq -1)}}^{\infty} \frac{a_n \bar{\tau}_j^{n+1}}{n+1} - a_{-1} \ln \frac{\tau_j}{\rho_j} + \mathrm{const} \right) \tag{6.8}$$

It can easily be seen that

$$a_{-1} = -\frac{1}{2(\varkappa+1)} Z_0^{(1)} = \frac{1}{2(\varkappa+1)} Z_0^{(2)} \tag{6.9}$$

and the constant appearing in (6.8) can be absorbed into $C^{(j)}$

Let us multiply both sides of (6.5) by $d\tau / (\tau - \zeta)$ and integrate over the contours L_1 and L_2. When the point ζ falls outside D, the function ψ is eliminated and we find

$$\sum_{n=-\infty}^{\infty} \frac{n-1}{2n-3} \alpha_n a_n \zeta^{n-2} + \sum_{n=-\infty}^{\infty} \frac{n}{2n-1} \beta_n a_{-n} \zeta^{n-1} = \sum_{n=-\infty}^{\infty} \frac{n+1}{2n+1} \Delta_n \zeta^n \tag{6.10}$$

where

$$\beta_n = \left(2\lambda_1 + \frac{2}{2n+1} - \frac{\varkappa - \lambda_1}{n-1}\right) \rho_1^{-2n+1} - \left(2\lambda_2 + \frac{2}{2n+1} - \frac{\varkappa - \lambda_2}{n-1}\right) \rho_2^{-2n+1} \qquad (n \neq 1)$$

$$\beta_1 = (\lambda_1 + {}^2/_3 + \varkappa)\, \rho_1^{-1} - (\lambda_2 + {}^2/_3 + \varkappa)\, \rho_2^{-1}$$

$$\alpha_n = 2n \frac{2n-3}{2n-1} (\rho_2^2 - \rho_1^2), \qquad \Delta_n = F_n^{(1)} - F_n^{(2)} \tag{6.11}$$

From (6.10) we easily obtain

$$a_n = \frac{\alpha_{-n+1} \Delta_{n-2} - \beta_{n-1} \Delta_{-n-1}}{\alpha_n \alpha_{-n+1} - \beta_{n-1} \beta_{-n}} \qquad (n \neq 0) \tag{6.12}$$

the coefficient a_0 remaining undefined.

If the external forces are given on any of the surfaces, then the coefficient a_{-1} can be found more conveniently with the aid of (6.9), in which case we have

$$a_2 = (\Delta_{-3} - \alpha_{-1} a_{-1}) / \beta_{-2} \tag{6.13}$$

and the constants $C^{(j)}$ need not be determined.

The function $\psi(\zeta)$ can be found in a similar manner, but ζ must in this case be situated within D. Performing the necessary manipulations, we obtain

$$b_n = -\frac{n+1}{2n+1} F_n^{(j)} - \frac{n-\varkappa}{2n+1} a_n - 2\rho_j^2 \frac{(n+1)(n+2)}{2n+3} a_{n+2} +$$

$$+ \left[\lambda_j + \frac{(2n+3)\varkappa + 2(n+1)}{(2n+1)(2n+3)}\right] a_{-n-1} \rho_j^{-2n-1} - \frac{\varkappa - \lambda_j}{n} a_{-n-1} \rho_j^{-2n-1} \tag{6.14}$$

($j = 1$ when $n \geqslant 0$, $j = 2$ when $n < 0$ and the last term is omitted, when $n = 0$).

When the functions $\varphi(\zeta)$ and $\psi(\zeta)$ are known, the displacements of the body points are easily obtained from (1.1), provided that we take into account

$$\frac{1}{\pi i}\int_{\bar t}^{t}\frac{\zeta^n d\zeta}{g(t,\zeta)}=\rho^n P_m(\mu), \qquad -\frac{1}{2\pi i r}\int_{\bar t}^{t}\zeta^n g_1(t,\zeta)\,d\zeta=\frac{\sqrt{1-\mu^2}}{n+1}\rho^n P_m{}'(\mu) \quad (6.15)$$

Here $m = n$ for $n \geqslant 0$, $m = |n| - 1$ for $n < 0$, $\rho = \sqrt{z^2 + r^2}$, $\mu = z / \rho$.
When a uniform pressure p_1 acts on L_1 and p_2 on L_2 (Lamé problem), we have

$$p_z^{(j)} = (-1)^j p_j \cos\vartheta, \quad p_r^{(j)} = (-1)^j p_j \sin\vartheta, \quad Z^{(j)} = -{}^1/_2(1-\mu^2)\rho_j p_j P_1{}'(\mu)$$

$$R^{(j)} = {}^1/_2\rho_j p_j\,[P_1(\mu) - P_0(\mu)], \qquad F_n^{(j)} = \Delta_n = 0 \qquad (n \neq 0, -2)$$

$$F_{-2}^{(j)} = {}^3/_2\rho_j{}^3 p_j, \qquad \Delta_{-2} = {}^3/_2(\rho_1{}^3 p_1 - \rho_2{}^3 p_2), \qquad \lambda_j = -1 \qquad (j = 1; 2)$$

$$a_n = 0 \quad (n \neq 1), \quad b_n = 0 \quad (n \neq 1, -2), \quad b_1 = {}^1/_3(\varkappa - 1)\,a_1$$

$$a_1 = \frac{\Delta_{-2}}{\beta_{-1}} = \frac{3}{4(1+\nu)}\,\frac{\rho_1{}^3 p_1 - \rho_2{}^3 p_2}{\rho_2{}^3 - \rho_1{}^3}, \qquad b_{-2} = {}^1/_2\rho_1{}^3\rho_2{}^3\,\frac{p_1 - p_2}{\rho_1{}^3 - \rho_2{}^3}$$

$$2G(w - iu) = [{}^2/_3(\varkappa - 1)\,a_1 - b_{-2}\rho^{-3}]\,(z - ir)$$

which coincides with the known solution.

Formulas analogous to (6.12) and (6.14) were obtained by another method in [7].

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